DSP

Chapter-9 : Filter Bank Design

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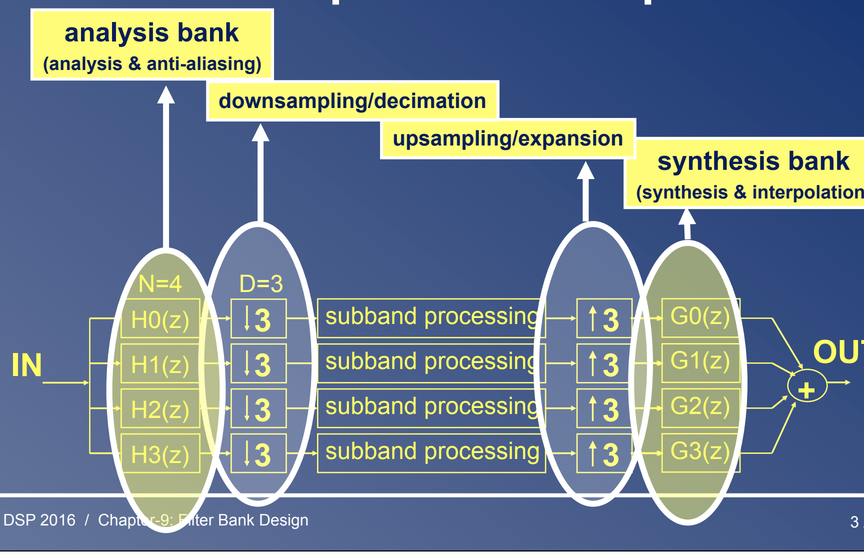
Part-IV : Filter Banks & Subband Systems

Chapter-8 **Filter Bank Preliminaries**
- Filter Bank Set-Up
- Filter Bank Applications
- Ideal Filter Bank Operation
- Non-Ideal Filter Banks: Perfect Reconstruction Theory

Chapter-9 **Filter Bank Design**
- Non-Ideal Filter Banks: Perfect Reconstruction Theory (continued)
- Filter Bank Design Problem Statement
- General Perfect Reconstruction Filter Bank Design
- DFT-Modulated Filter Banks

Filter Bank Set-Up

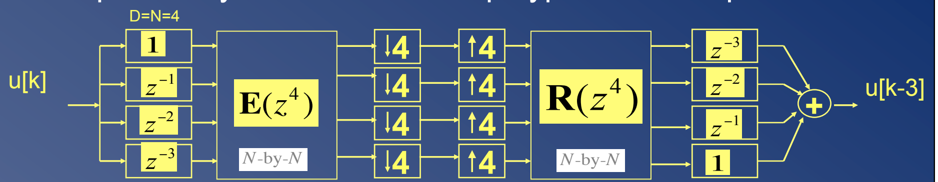
So this is the picture to keep in mind...



Perfect Reconstruction Theory (continued)

$D = N$

A simpler analysis results from a polyphase description :



$$\begin{bmatrix} H_0(z)\\ \vdots \\ H_{N-1}(z)\end{bmatrix} = \overbrace{\begin{bmatrix} E_{0|0}(z^N) & \dots & E_{0|N-1}(z^N)\\ \vdots & & \vdots\\ E_{N-1|0}(z^N) & \dots & E_{N-1|N-1}(z^N)\end{bmatrix}}^{\mathbf{E}(z^N)} \cdot \begin{bmatrix}1\\ \vdots\\ z^{-(N-1)}\end{bmatrix}$$

n-th row of E(z) has N-fold (=D-fold) polyphase components of Hn(z)

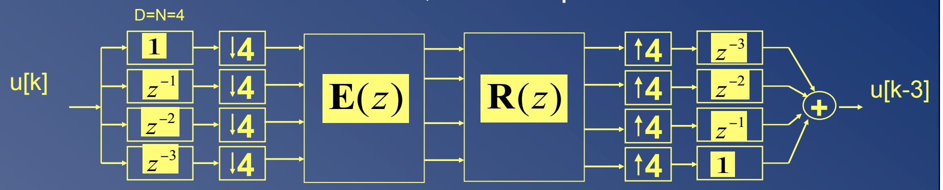
$$\begin{bmatrix} F_0(z)\\ \vdots \\ F_{N-1}(z)\end{bmatrix}^T = \begin{bmatrix}z^{-(N-1)}\\ \vdots\\ 1\end{bmatrix}^T \cdot \overbrace{\begin{bmatrix} R_{0|0}(z^N) & \dots & R_{N-1|0}(z^N)\\ \vdots & & \vdots\\ R_{0|N-1}(z^N) & \dots & R_{N-1|N-1}(z^N)\end{bmatrix}}^{\mathbf{R}(z^N)}$$

n-th column of R(z) has N-fold polyphase components of Fn(z)

Do not continue until you understand how formulae correspond to block scheme!

Perfect Reconstruction Theory

- With the `noble identities', this is equivalent to:



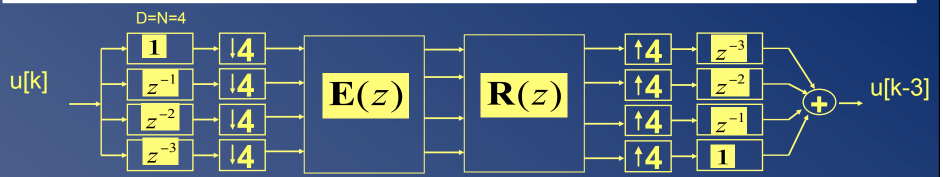
Necessary & sufficient conditions for

i) alias cancellation

ii) perfect reconstruction

are then derived, based on the product $\mathbf{R}(z).\mathbf{E}(z)$

Perfect Reconstruction Theory



Necessary & sufficient condition for **PR** is then (proof omitted)…

$$\mathbf{R}(z).\mathbf{E}(z) = \begin{bmatrix} 0 & z^{-\delta} I_{N-r} \\ z^{-\delta-1}.I_r & 0 \end{bmatrix}, \quad 0 \le r \le N-1$$

I_n is nxn identity matrix, r is arbitrary

Example (r=0) : $\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_N$ for conciseness, will use this from now on !

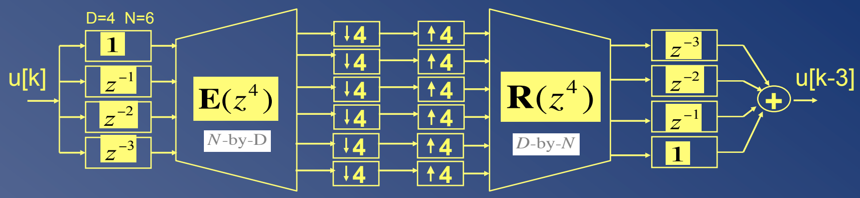
Beautifully simple!!

Perfect Reconstruction Theory

$D < N$

A similar PR condition can be derived for oversampled FBs

The polyphase description (compare to p.34) is then…



$$\begin{bmatrix} H_0(z) \\ \vdots \\ H_{N-1}(z)\end{bmatrix} = \overbrace{\begin{bmatrix} E_{0|0}(z^D) & \ldots & E_{0|D-1}(z^D) \\ \vdots & & \vdots \\ E_{N-1|0}(z^D) & \ldots & E_{N-1|D-1}(z^D)\end{bmatrix}}^{\mathbf{E}(z^D)} \cdot \begin{bmatrix} 1 \\ \vdots \\ z^{-(D-1)}\end{bmatrix}$$

n-th row of E(z) has D-fold polyphase components of $H_n(z)$

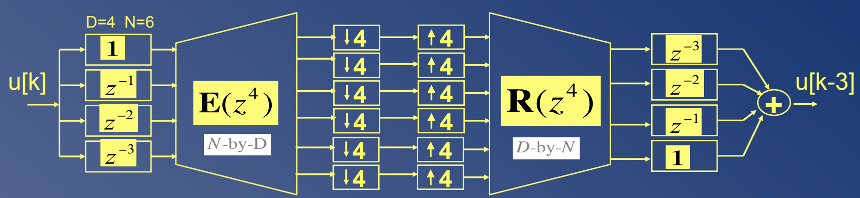
$$\begin{bmatrix} F_0(z) \\ \vdots \\ F_{N-1}(z)\end{bmatrix}^T = \begin{bmatrix} z^{-(D-1)} \\ \vdots \\ 1\end{bmatrix}^T \cdot \overbrace{\begin{bmatrix} R_{0|0}(z^D) & \ldots & R_{N-1|0}(z^D) \\ \vdots & & \vdots \\ R_{0|D-1}(z^D) & \ldots & R_{N-1|D-1}(z^D)\end{bmatrix}}^{\mathbf{R}(z^D)}$$

n-th column of R(z) has D-fold polyphase components of $F_n(z)$

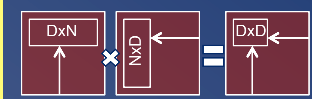
Note that **E** is an N-by-D ('tall-thin') matrix, **R** is a D-by-N ('short-fat') matrix !

Perfect Reconstruction Theory

Simplified (cfr. r=0 on p.6) condition for **PR** is then…



$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_D$$



In the D=N case (p.6), the PR condition has a product of square matrices. PR-FB design will then involve matrix inversion, which is mostly problematic.

In the D<N case, the PR condition has a product of a 'short-fat' matrix and a 'tall-thin' matrix. This will lead to additional PR-FB design flexibility.

Again beautifully simple!!

Filter Bank Design Problem Statement

Two design targets :

- ✪ **Filter specifications**, e.g. stopband attenuation, passband ripple, transition band, etc. (for each (analysis) filter!)
- ✪ **Perfect reconstruction** (PR) property.

Challenge will be in addressing two design targets at once

(e.g. 'PR only' (without filter specs) is easy, see ex. Chapter-8)

PS: Can also do 'Near-Perfect Reconstruction Filter Bank Design', i.e. optimize filter specifications and at the same time minimize aliasing/ distortion (=numerical optimization). Not covered here…

General PR-FB Design: Maximum Decimation (D=N)

$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_N$$

(= N-by-N matrices)

- Design Procedure:
  1. Design all analysis filters (see Part-II).
  2. This determines E(z) (=polyphase matrix).
  3. Assuming E(z) can be inverted (?), synthesis filters are

$$\mathbf{R}(z) = z^{-\delta}.\mathbf{E}^{-1}(z)$$

(delta to make synthesis causal, see ex. p.7)

- Will consider only **FIR** analysis filters, leading to simple polyphase decompositions (see Chapter-2)
- However, FIR E(z) then generally leads to **IIR** R(z), where stability is a concern…

General PR-FB Design: Maximum Decimation (D=N)

PS: Inversion of matrix transfer functions ?…

- The inverse of a scalar (i.e. 1-by-1 matrix) FIR transfer function is **always** IIR (except for contrived examples)

$$\mathbf{E}(z) = (2 - z^{-1}) \Rightarrow \mathbf{R}(z) = \mathbf{E}^{-1}(z) = \frac{1}{(2 - z^{-1})}$$

- …but the inverse of an N-by-N (N>1) FIR transfer function **can** be FIR

$$\mathbf{E}(z) = \begin{bmatrix} z^{-2} + \frac{1}{2} & z^{-1} \\ 2z^{-1} & 2 \end{bmatrix} \Rightarrow \mathbf{R}(z) = \mathbf{E}^{-1}(z) = \begin{bmatrix} 2 & -z^{-1} \\ -2z^{-1} & z^{-2} + \frac{1}{2} \end{bmatrix}$$

$$\det(\mathbf{E}(z)) = 1$$

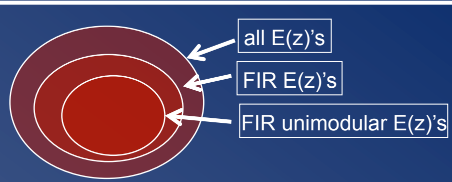
PS: Compare this to inversion of integers and integer matrices

$\mathbf{E} = 2 \Rightarrow \mathbf{R} = \mathbf{E}^{-1} = \frac{1}{2}$ …but… $\mathbf{E} = \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix} \Rightarrow \mathbf{R} = \mathbf{E}^{-1} = \begin{bmatrix} 5 & -6 \\ -4 & 5 \end{bmatrix}$, $\det(\mathbf{E}) = 1$

General PR-FB Design: Maximum Decimation (D=N)

Question:

Can we build **FIR** E(z)'s (N-by-N) that have an **FIR** inverse?



Answer:

YES, `**unimodular**' E(z)'s, i.e. matrices with determinant=constant*z^d

e.g.

$$\mathbf{E}(z) = \mathbf{E}_L . \begin{bmatrix} I_{N-1} & 0 \\ 0 & z^{-1} \end{bmatrix} . \mathbf{E}_{L-1} . \begin{bmatrix} I_{N-1} & 0 \\ 0 & z^{-1} \end{bmatrix} \ldots \mathbf{E}_1 . \begin{bmatrix} I_{N-1} & 0 \\ 0 & z^{-1} \end{bmatrix} . \mathbf{E}_0$$

$$\mathbf{R}(z) = \mathbf{E}_0^{-1} . \begin{bmatrix} z^{-1}.I_{N-1} & 0 \\ 0 & 1 \end{bmatrix} . \mathbf{E}_1^{-1} \ldots \begin{bmatrix} z^{-1}.I_{N-1} & 0 \\ 0 & 1 \end{bmatrix} . \mathbf{E}_{L-1}^{-1} . \begin{bmatrix} z^{-1}.I_{N-1} & 0 \\ 0 & 1 \end{bmatrix} . \mathbf{E}_L^{-1}$$

$$\Rightarrow \mathbf{R}(z).\mathbf{E}(z) = z^{-L}.I_N$$

where the E_i's are constant (= not a function of z) invertible matrices

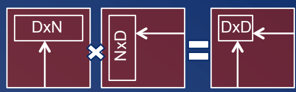
Design Procedure:

Optimize E_i's to meet filter specs (ripple, etc.) for all analysis filters (at once)

= not-so-easy but DOABLE !

General PR-FB Design: Oversampled FBs (D<N)

$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_D$$



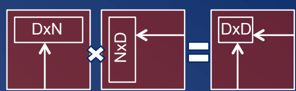
- <u>Design Procedure:</u>
  1. Design all analysis filters (see Part-II).
  2. This determines E(z) (=polyphase matrix).
  3. Find R(z) such that PR condition is satisfied (how? read on…)

= easy if step-3 is doable…

- Will consider only **FIR** analysis filters, leading to simple polyphase decompositions (see Chapter-2)
- It will turn out that when D<N an **FIR** R(z) can always be found (except in contrived cases)…

General PR-FB Design: Oversampled FBs (D<N)

$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_D$$



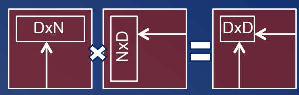
- <u>Given E(z) how can R(z) be computed?</u>
  - Assume every entry in E(z) is L_E-th order FIR (i.e. L_E +1 coefficients)
  - Assume every entry in R(z) is L_R-th order FIR (i.e. L_R +1 coefficients)
  - Hence number of unknown coefficients in R(z) is **$D.N.(L_R + 1)$**
  - Every entry in R(z).E(z) is (L_E+L_R)-th order FIR (i.e. L_E+L_R+1 coefficients) (cfr. polynomial multiplication / linear convolution)
  - Hence PR condition is equivalent to **$D.D.(L_E+L_R+1)$** linear equations in the unknown coefficients (*)
  - Can be solved (except in contrived cases) if $D.N.(L_R+1) \geq D.D.(L_E+L_R+1)$

$$\rightarrow \quad L_R \geq \frac{D}{(N-D)} L_E - 1$$

(*) Try to write down these equations!

General PR-FB Design: Oversampled FBs (D<N)

$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_D$$



- Given E(z) how can R(z) be computed?
  - (continued) …
  - Can be solved (except in contrived cases) if $D.N.(L_R+1) \geq D.D.(L_E+L_R+1)$

    ➔ $L_R \geq \frac{D}{(N-D)} L_E - 1$
  - If D<N, then L_R can be made sufficiently large so that the (underdetermined) set of equations can be solved, i.e. an R(z) can be found (!).
  - Note that if D=N, then L_R in general has to be infinitely large, i.e. R(z) in general has to be IIR

DFT-Modulated FBs

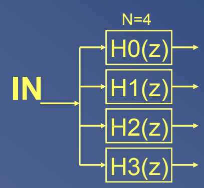
- All design procedures so far involve monitoring of characteristics (passband ripple, stopband suppression,…) of all (analysis) filters, which may be tedious.

- Design complexity may be reduced through usage of \`uniform' and \`modulated' filter banks.
  - DFT-modulated FBs (read on..)
  - Cosine-modulated FBs (not covered, but interesting design!)

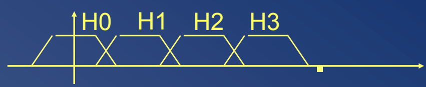
- Will consider
  - Maximally decimated DFT-modulated FBs
  - Oversampled DFT-modulated FBs

Maximally Decimated DFT-Modulated FBs (D=N)

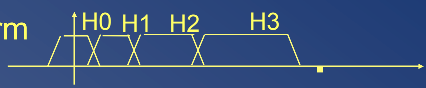
Uniform versus non-uniform (analysis) filter bank:



uniform



non-uniform



- **N-channel uniform FB**: $H_n(z) = H_0(z.e^{-j2\pi n/N}) \quad n = 0,\ldots,N-1$

  i.e. frequency responses are uniformly shifted over the unit circle

  Ho(z)= \`prototype' filter (=one and only filter that has to be designed)

  Time domain equivalent is: $h_n[k] = h_0[k].e^{j2\pi k.n/N}$

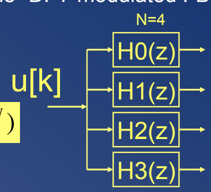
- Non-uniform = everything that is not uniform

  e.g. for speech & audio applications (cfr. human hearing)

Maximally Decimated DFT-Modulated FBs (D=N)

Uniform filter banks can be realized cheaply based on polyphase decompositions + DFT(FFT) (hence name \`DFT-modulated FB)

1. Analysis FB



If $H_0(z), H_1(z), \ldots, H_{N-1}(z) \quad \text{with} \quad H_n(z) = H_0(z.e^{-j2\pi n/N})$

$$H_0(z) = \sum_{\bar{n}=0}^{N-1} z^{-\bar{n}}.E_{\bar{n}}(z^N)$$

(N-fold polyphase decomposition)

then

$$H_n(z) = H_0(z.e^{-j2\pi n/N}) = \sum_{\bar{n}=0}^{N-1} z^{-\bar{n}}.e^{j2\pi n\bar{n}/N}.E_{\bar{n}}(z^N \overbrace{e^{-j2\pi nN/N}}^{1})$$

$$= \sum_{\bar{n}=0}^{N-1} z^{-\bar{n}}.W^{-n\bar{n}}.E_{\bar{n}}(z^N) , \quad \text{with} \quad W = e^{-j2\pi/N}$$

i.e. →

Maximally Decimated DFT-Modulated FBs (D=N)

i.e. →

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ \vdots \\ H_{N-1}(z) \end{bmatrix} . U(z) = \overbrace{\begin{bmatrix} W^0 & W^0 & W^0 & \dots & W^0 \\ W^0 & W^{-1} & W^{-2} & \dots & W^{-(N-1)} \\ W^0 & W^{-2} & W^{-4} & \dots & W^{-2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W^0 & W^{-(N-1)} & W^{-2(N-1)} & \dots & W^{-(N-1)^2} \end{bmatrix}}^{N.\mathbf{F}^{-1}} . \begin{bmatrix} E_0(z^N) \\ z^{-1}.E_1(z^N) \\ z^{-2}.E_2(z^N) \\ \vdots \\ z^{-N+1}.E_{N-1}(z^N) \end{bmatrix} . U(z)$$

$$W = e^{-j2\pi/N}$$

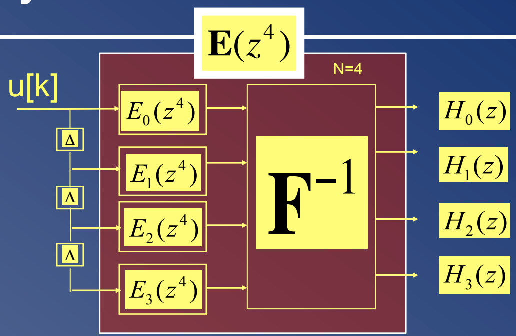
where F is NxN DFT-matrix

This means that filtering with the H_n's can be implemented by first filtering with the polyphase components and then applying an inverse DFT

PS: To simplify formulas the factor N in $N.F^{-1}$ will be left out from now on (i.e. absorbed in the polyphase components)

Maximally Decimated DFT-Modulated FBs (D=N)

i.e. →

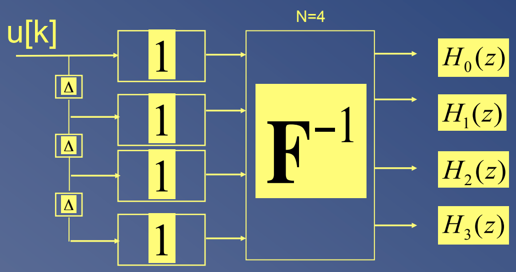


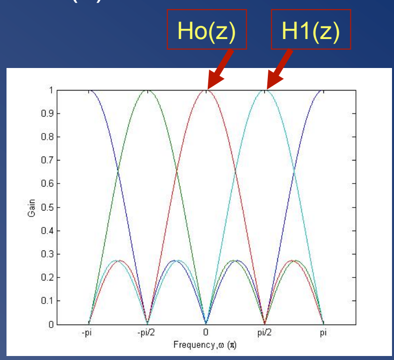
Conclusion: economy in…

- Implementation complexity (for FIR filters):
  N filters for the price of 1, plus inverse DFT (=FFT) !
- Design complexity:
  Design \`prototype' Ho(z), then other $H_n(z)$'s are automatically \`co-designed' (same passband ripple, etc…) !

Maximally Decimated DFT-Modulated FBs (D=N)

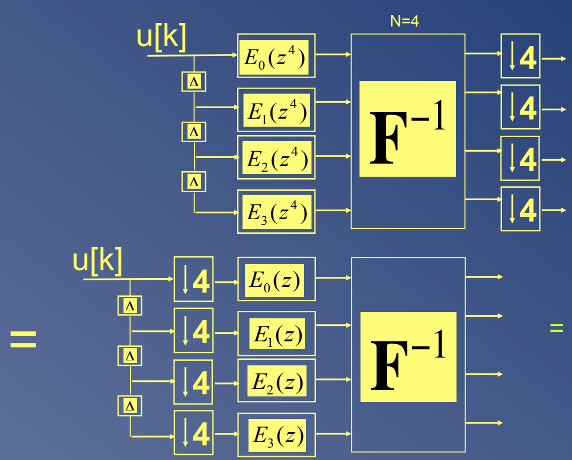
- Special case: DFT-filter bank, if all $E_n(z)=1$





Maximally Decimated DFT-Modulated FBs (D=N)

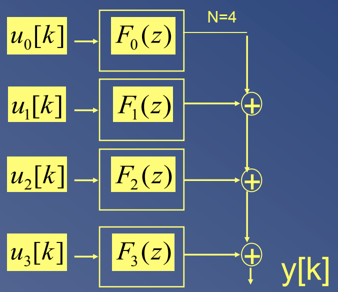
- DFT-modulated analysis FB + maximal decimation



= efficient realization !

Maximally Decimated DFT-Modulated FBs (D=N)

2. Synthesis FB



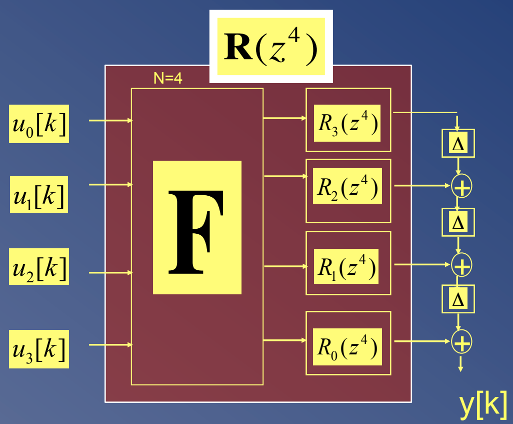
phase shift added for convenience

$$F_0(z), F_1(z), \ldots, F_{N-1}(z) \quad \text{with} \quad F_n(z) = e^{j2\pi n/N} . F_0(z.e^{-j2\pi n/N})$$

$$F_0(z) = \sum_{\bar{n}=0}^{N-1} z^{-\bar{n}} . R_{\bar{n}}(z^N) \quad \Rightarrow \quad F_n(z) = \quad \ldots = \sum_{\bar{n}=0}^{N-1} z^{-\bar{n}} . W^{n(N-1-\bar{n})} . R_{\bar{n}}(z^N)$$

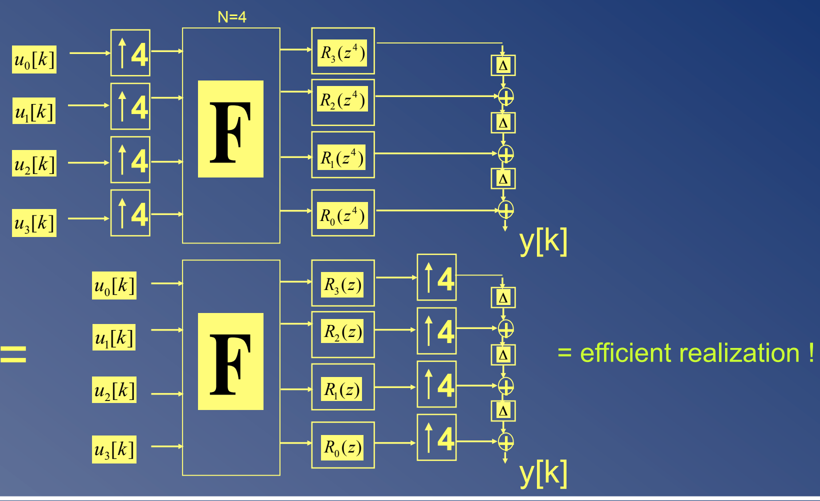
Maximally Decimated DFT-Modulated FBs (D=N)

Similarly simple derivation then leads to…



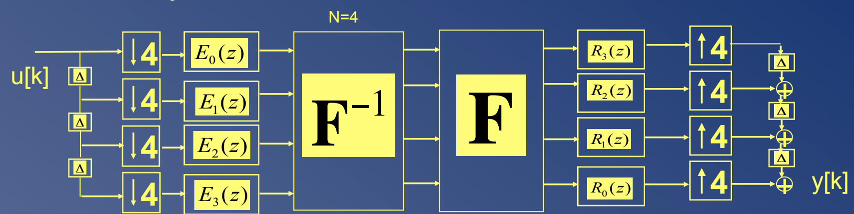
Maximally Decimated DFT-Modulated FBs (D=N)

- Expansion + DFT-modulated synthesis FB :



Maximally Decimated DFT-Modulated FBs (D=N)

How to achieve **Perfect Reconstruction** (PR)
with maximally decimated DFT-modulated FBs?



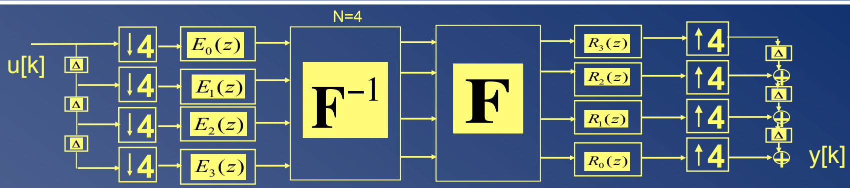
$$\mathbf{R}(z).\mathbf{E}(z) = z^{-\delta} I_N$$

$$\rightarrow \mathbf{E}(z)=\mathbf{F}^{-1}.diag[E_n(z)] \Rightarrow \mathbf{R}(z)=z^{-\delta}.\mathbf{E}^{-1}(z)=z^{-\delta}.diag[E_n^{-1}(z)].\mathbf{F}$$

$$\rightarrow R_n(z)=z^{-\delta}.E^{-1}_{N-1-n}(z)$$

Polyphase components of synthesis bank prototype filter are obtained by inverting polyphase components of analysis bank prototype filter

Maximally Decimated DFT-Modulated FBs (D=N)



- Design Procedure:
  1. Design prototype analysis filter Ho(z) (see Part-II).
  2. This determines $E_n(z)$ (=polyphase components).
  3. Assuming all $E_n(z)$'s can be inverted (?), choose synthesis filters

$$R_n(z) = z^{-\delta}.E^{-1}_{N-1-n}(z)$$

- Will consider only **FIR** prototype analysis filter, leading to simple polyphase decomposition (Chapter-2).
- However, FIR $E_n(z)$'s generally again lead to IIR $R_n(z)$'s, where stability is a concern…

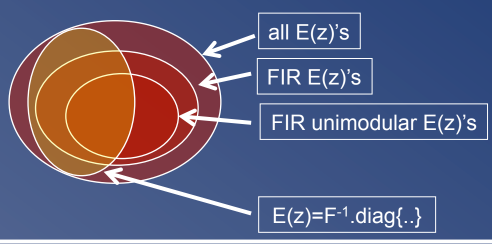
Maximally Decimated DFT-Modulated FBs (D=N)

This does not leave much design freedom…

- **FIR E(Z)?** ..such that Rn(z) are also FIR

  Only obtained when each $E_n(z)$ is 'unimodular', i.e. $E_n(z)$=constant.z^d

  Simple example is $E_n(z) = w_n \Rightarrow R_{N-1-n}(z) = w_n^{-1}$, where w_n's are constants, which leads to \`windowed' IDFT/DFT bank, a.k.a. \`short-time Fourier transform' (see Chapter-14)



Maximally Decimated DFT-Modulated FBs (D=N)

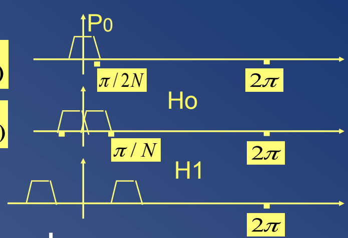
- **Bad news:** Not much design freedom for maximally decimated DFT-modulated FB's…
- **Good news:** More design freedom with...
  - **Cosine-modulated** FB's
  - **Oversampled** DFT-modulated FB's

Cosine Modulated FBs

- Procedure:
  Po(z) = prototype lowpass filter, cutoff at $\pm\pi/2N$ for N filters
  Then...

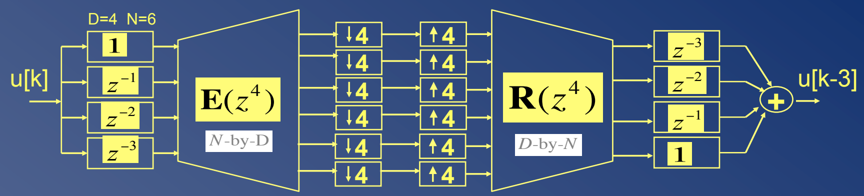
$$H_0(z) = \alpha_0 . P_0(z.e^{-j(0.5)\frac{\pi}{N}}) + \alpha_0^* . P_0(z.e^{j(+0.5)\frac{\pi}{N}})$$

$$H_1(z) = \alpha_1 . P_0(z.e^{-j(1+0.5)\frac{\pi}{N}}) + \alpha_1^* . P_0(z.e^{j(1+0.5)\frac{\pi}{N}})$$



  etc..
  PS: Real-valued filter coefficients here!
- Details: See literature…
- Maximally decimated & oversampled FB designs
- Design software available (e.g.Matlab)

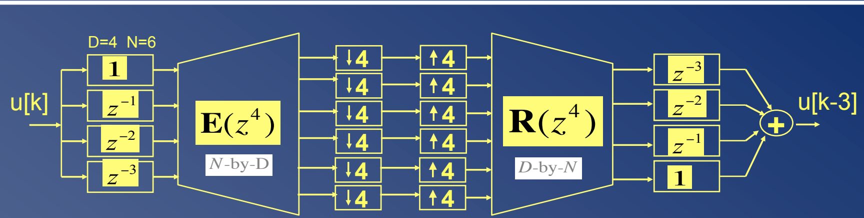
Oversampled DFT-Modulated FBs (D<N)



- In maximally decimated DFT-modulated FB, we had
  $\mathbf{E}(z)=\mathbf{F}^{-1}.diag\left[E_n(z)\right]$ $\mathbf{R}(z)=diag\left[R_{N-1-n}(z)\right].\mathbf{F}$ (N-by-N matrices)
- In oversampled DFT-modulated FB, will have
  $\underbrace{\mathbf{E}(z)}_{\text{N-by-D}}=\underbrace{\mathbf{F}^{-1}}_{\text{N-by-N}}.\underbrace{\mathbf{B}(z)}_{\text{N-by-D}}$ $\underbrace{\mathbf{R}(z)}_{\text{D-by-N}}=\underbrace{\mathbf{C}(z)}_{\text{D-by-N}}.\underbrace{\mathbf{F}}_{\text{N-by-N}}$

  with B(z) (tall-thin) and C(z) (short-fat) structured/sparse matrices constructed with poyphase components of prototype filters

Oversampled DFT-Modulated FBs (D<N)



- Details: see literature
- Design Examples:
  http://homes.esat.kuleuven.be/~dspuser/DSP-CIS/2016-2017/material.html
- Design software available (e.g.Matlab)